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A CONVERGENT DEMAND REVEALING PROCESS

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ABSTRACT

Presented is a decentralized tâtonnement public good allocation process that converges to an efficient allocation in a wide class of two good neoclassical economies when consumers are permitted to be non-competitive utility maximizers. The process is a demand revealing mechanism generalized to include an enforcement rule administered by the government which defines a response function, making the government a dynamically active participant in the allocation process. This generalization greatly improves upon the basic mechanism which has been considered extensively in the literature, and under which the government functions as a passive public good procurement agent when the model is formulated dynamically, solely processing information received from consumers and then implementing the implied demands. Inclusion of an enforcement rule permits the government to impose an intertemporal consistency requirement on the communications of individual consumers. It is shown that there is an enforcement rule so that a strategy equilibrium is always attained asymptotically and so that the equilibrium allocation is efficient; in equilibrium, there is no waste, no bankruptcy and consumers communicate local truth.

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1. INTRODUCTION

Considerable attention has been focused recently on the possibility of designing social choice mechanisms such that an equilibrium under these mechanisms will satisfy given social performance criteria. In particular, efforts have been directed at the design of resource allocation mechanisms relative to which equilibrium allocations are Pareto efficient for a wide class of economic environments.[1] A seminal paper in this literature is that of Groves and Ledyard (1977) in which a specific mechanism--formulated as a system of allocation and taxing rules--is proposed that yields Pareto efficient equilibrium allocations of resources to the production of public goods in a model where consumers are permitted to misrepresent their true preferences if they wish. The significance of the Groves-Ledyard contribution and the demand revelation literature in general is diminished only by the general non-implementability of the mechanisms that have been proposed. The major obstacles to the theoretical implementability of the mechanisms are the consumer bankruptcy problem and the problem of ensuring that the mechanisms will dynamically generate an equilibrium when consumers' utilities do not satisfy a stringent separability condition.[2] Unless some procedure can be demonstrated that circumvents these difficulties while maintaining the desirable properties of the demand revelation mechanisms, the approach is not of practical significance. The purpose of this paper is to develop a procedure that implements a demand revelation mechanism for a wide class of neoclassical economic environments.

Previous investigations of the implementability question have concentrated on the stability of equilibria for particular economic environments relative to mechanisms with the Groves-Ledyard (G-L) allocation rule that are administered by a government under alternative dynamic behavioral hypotheses.[3] Each hypothesized behavior for agents defines a system of response rules for the agents which, in turn, completely determines the economic system's adjustment dynamics. If equilibria for the posed economic system are stable for a given set of agents' response rules, then the allocation mechanism generates (i.e. converges to) an equilibrium, and, in this sense, is implementable provided agents behave as assumed. In general, the literature has viewed these models as N-player non-cooperative games, and no reasonable dynamic behavioral hypothesis has been shown to yield stable strategy equilibria within this basic framework for any general economic environment.[4] For example, Bamford (1979) and Ledyard (1978) both have to restrict severely the values of various utility parameters for every specific economic environment considered to obtain stability or convergence under a Cournot behavioral hypothesis. Ledyard found the instability to be due to bad income effects or overreactions by agents. To obtain convergence under alternative behavioral hypotheses, Ledyard (1978) has to assume there are no income effects. The consumer bankruptcy problem is not addressed.

The role of the government in these models has always been dynamically passive. Once the government has announced the allocation and tax rules and the allowable message space, its sole responsibility has been to process the information received at each iteration according to these rules, and then to transmit the results back to consumers, terminating the process according to some stopping rule.

In this paper, the general iterative or dynamic game theoretic approach to the implementation problem is not emphasized; rather, the problem is viewed as principally a question of optimal institutional design. By exploiting the

notion that an enforcement structure is implicit in any specification of a demand revelation mechanism, we are able to develop a model in which the government's role is not dynamically passive. The government is permitted _to learn_ from history (i.e. previous adjustments), and to enforce this history by restricting subsequent adjustments in whatever manner it believes to be consistent with the history--a notion that requires definition--while still allowing consumers considerable autonomy in their decisions. This learning mechanism for government entails a technically slight but conceptually significant modification of the demand revelation mechanism. Here, the economic system's adjustment dynamics is determined by an evolving enforcement structure--the government's response rule--_as well as_ consumers' response rules.

The adjustments are iterative, the procedure is tâtonnement, and the model is deterministic. Thus, the appropriate concept of equilibrium is that of _best replay_ where consumers communicate successively _equivalent_ strategies and the government does not alter the enforcement structure (the set of time-specific permissible strategies or communications). When viewed statically, this equilibrium concept is a straightforward generalization of the Nash equilibrium as employed by Groves-Ledyard (1977) and others. The general formal structure of the model for two good (one private, one public) neo-classical economies is presented in the next section.

In section 3, a particular tâtonnement allocation process is considered. For this process, consumers are required to communicate entire functions that satisfy a uniform curvature restriction. For each vector of functions communicated by consumers, the government selects a single point for each consumer, using the allocation rule, and then restricts consumers' subsequent communications to the subclass of allowable functions that pass through the individually assigned points, and so on. Thus, the enforcement structure evolves gradually by restricting further with each round of communications the choice sets for consumers.

What is important is that the particular nature of the intertemporal collapse of each consumer's choice set is determined by the strategic interactions of the consumers themselves, given the allocation and tax rules. It is not predetermined by the government's enforcement rule. The guiding principle of the procedure is that the government treats consumers' communications as if they are truthful representations of individual preferences. A consumer is penalized for attempted manipulations only in so far as they may result in restrictions on his ability to counter the subsequent reactions of other consumers. No direct costs are imposed by the government for attempted manipulations.

For a wide class of neoclassical economies, this process always generates an asymptotic, first order Pareto efficient allocation of resources if consumers' behaviors satisfy a Cournot hypothesis. Moreover, the process always converges, regardless of how consumers behave, although the efficiency properties of the asymptotic equilibrium may be lost if consumers do not act as "rational" or consistent utility maximizing players. This convergence property is very much a consequence of the government's response rule, and therefore the dynamic nature of the enforcement structure is an integral part of the institutional design. The significance of the convergence property is two-fold: (1) overreactions and income effects, though permitted initially, are eliminated asymptotically, and (2) no consumer can be bankrupted in equilibrium if he only communicates financially feasible strategies at each iteration. The government's budget can always be balanced in equilibrium.[5] Finally, the mechanism is individually incentive compatible (in a qualitative

sense). In section 4, we offer some criticisms of our results that are motivated by both practical and theoretical considerations, and we suggest some avenues for extension. Proofs of the technical lemmas cited in the text are relegated to an appendix.

2. THE MODEL

The economic system considered here is a simple two good neoclassical economy with a single public good and a government. The government's only objective (by assumption) is to administer a well-defined program that provides for a Pareto efficient equilibrium allocation of resources to the production of the public good, financing this activity by taxing consumers. It is assumed further that the government has no apriori knowledge of consumers' preferences so that the government's primary problem is to elicit sufficient information from consumers to solve the efficiency problem. Ideally, each consumer would respond "truthfully" to the government's request for demand information, but it is well-known that this ideal is not attained (the free rider problem) unless it is in the consumer's self-interest to communicate truth. To obtain efficient, possibly untruthful, information is a less formidable objective, although the revelation problem is still non-trivial.[6] To accomplish its objective, the government must adopt a program that is implementable; the program must provide for an adjustment procedure that ensures convergence to an efficient equilibrium even though individual consumers may try to exploit information they receive to manipulate the outcome in their favor. In short, the program must include an intertemporal enforcement or punishment element that makes it irrational for consumers to continue to communicate inefficient, manipulative information even though any one misrepresentation may not be detectable at the time it is communicated.

2.1 The Economy

Formally, assume there are two commodities, a pure private good x and a pure public good y, N consumers (indexed $i \leq N$), a single producer and a government. The commodity space is $\mathbb{R}^2_+$. Each consumer is characterized by a

utility function $u^i : \mathbb{R}^2_+ \rightarrow \mathbb{R}$, and an initial endowment of the private good, $w_i \in \mathbb{R}_+$; no agent holds an intial endowment of the public good y. The difference between the final and initial holdings of the private good by the ith consumer is denoted by a non-positive scalar z_i, and his consumption of the private good by $x_i = w_i + z_i$. It is assumed further that the private good x is an input used to produce y, and that constant returns to scale prevail. With suitable normalization, the input-output ratio can be set equal to one, so that the production equation becomes $\sum_{i \leq N} z_i + y = 0$.[7]

A state of the economy may be represented by an (N+1) - tuple $(x,y) = (x_i, \ldots, x_N; y) \in \mathbb{R}^{N+1}_+$, and the set of feasible states by the set $W = \{(x,y) \in \mathbb{R}^{N+1}_+ : \sum_{i \leq N} x_i + y = \sum_{i \leq N} w_i\}$. The economy can then be summarized by the list $e = \{(u^i, w_i, i \leq N), W\}$.[8]

<u>Assumption 1</u>. The economy e satisfies the following conditions: for each $i \leq N$,

(a) $w_i > 0$, and

(b) $u^i \in C^2(\mathbb{R}^2, \mathbb{R})$ such that $u^i(x_i,y) \geq u^i(\bar{x}_i,\bar{y})$ implies $u^i(x_i(\lambda),y(\lambda)) \geq u^i(\bar{x}_i,\bar{y})$ and $u^i(x_i,y) > u^i(\bar{x}_i,\bar{y})$ implies $u^i(x_i(\lambda),y(\lambda)) > u^i(\bar{x}_i,\bar{y})$ for $0 < \lambda < 1$ where $x_i(\lambda) = \lambda x_i + (1-\lambda)\bar{x}_i$ and $y(\lambda) = \lambda y + (1-\lambda)\bar{y}$.

2.2 The Government

The government's plan for determining the output level of the public good and the financing of this activity consists of an abstract message space, an allocation (or outcome) rule, individual tax rules, and an enforcement rule. The message space establishes the language for which the plan is defined; consumers must select the messages they communicate to the government from this space. The allocation and tax rules evaluate the messages transmitted by consumers and then specify, respectively, the allocation of resources to production of the public good and the individual consumer tax burdens. The enforcement rule stipulates how the allowable messages for each consumer are to be revised over time based on the previous position of the economy as given by the previous set of allowable messages and the consumers' message selection from this set.

When choosing an enforcement rule and an initial message space, the government may utilize any general economic information that it already has. We assume that the government knows the production equation and the equilibrium price ratio which is one, thereby avoiding the problem of price adjustment dynamics. Definition 1 formalizes the notion of the government's economic plan for provision of the public good.

Definition 1 Let A be an index set, and let $\mathcal{M} = \{M_\alpha \subset M : \alpha \in A\}$ be a collection of subsets of M. The economic plan administered by the government is given by a list $G = \{M, F, (C^i, i \leq N), E\}$ where

(a) $M = \underset{i \leq N}{\times} M_i$ is the initial message space;

(b) $F : M \to \mathbb{R}$, $F(m) = y$ is the allocation rule;

(c) $C^i : M \to \mathbb{R}$, $C^i(m) = c_i$, $i \leq N$ are the consumers' tax rules;

(d) $E : \mathcal{M} \times M \to \mathcal{M}$, $E(M_\alpha, m) = M_\beta$ for $\alpha, \beta \in A$ is the enforcement rule.

When selecting messages to send to the government, consumers may knowingly select messages that misrepresent their true preferences. However, the ability of a consumer to perpetrate a misrepresentation depends on whether it is detectable, and this in turn depends on the government's enforcement structure. One can think of the enforcement structure as being modeled implicitly in terms of the allowable misrepresentations. Thus, any restrictions on the set of permissible messages at a given point in time constitutes an implicit enforcement structure at that time. And, any rule that alters the restrictions on the set of permissible messages over time

defines a dynamic enforcement structure. Therefore, in Definition 1, a dynamic enforcement structure is given by the triple $(M, \mathcal{M}, E)$ where M is the universal or base message space, $\mathcal{M}$ is a collection of subsets of M, and E is a function that associates with each set of currently permissible messages $M_\alpha \in \mathcal{M}$ and each possible vector of individual messages selectable by consumers $m \in M_\alpha \subset M$, a set $M_\beta \in \mathcal{M}$ of permissible messages for the next iteration. If $M_{\alpha_{t+1}} \subset M_{\alpha_t}$ for all $t \geq 0$, $\alpha_t \in A$, then $E(\cdot)$ is said to define an evolving enforcement structure. We assume that any message selected from outside the permissible set by a consumer is immediately detectable by the government, and that such action results in a prohibitively large fine, effectively placing the action outside the consumer's choice set.

2.3 The Dynamics

For the model to include a well-defined dynamical system, additional structure must be imposed on both consumers' behaviors and the interaction between consumers and the government. In the system developed here, the government initiates the adjustment process by communicating to consumers an economic plan G and an initial message vector $m_o \in M$. Using this information, each consumer i responds with a message $m_{i1} \in M_{i1} = E_i(M, m_o)$, which the government then uses in the economic plan G to compute a tentative public good production y_1, individual tax burdens c_{i1}, $i \leq N$, and revised message spaces $M_{i2} = E_i(M_1, m_1)$ where $M_1 = \underset{i \leq N}{\times} M_{i1}$ and $m_1 = (m_{i1}, i \leq N)$. This completes the first iteration of the process. The tth iteration begins when the government informs consumers of the results of the (t-1)st iteration, viz. y_{t-1}, $(c_{i\,t-1}, i \leq N)$, m_{t-1} and $M_t = E(M_{t-1}, m_{t-1})$.

A consumer's best responses to the information he might receive from the government may be summarized in the form of a set-valued response rule:

$S^i : \{G\} \times \mathcal{M} \times M \rightarrow M_i$, $S^i(G, M_t, m_{t-1}) = \{m_{it}\} \subset M_{it}$. The particular characteristics of this response rule will be determined by the behavioral strategy selected by the consumer. Let $S = (S^i, i \leq N)$.

Definition 2. A dynamical system for an economy e relative to an economic plan G is given by a list $D = (M, \mathcal{M}, E, S)$. For each initial condition $m_o \in M$, a dynamical system D will be associated with two trajectories:

(a) a real trajectory given by the sequence, $\{(x_t, y_t), t \geq 1\}$ where $y_t = F(m_t)$ and $x_t = (x_{it}, i \leq N) = (w_i - c_{it}, i \leq N)$,

(b) a message trajectory given by the sequence, $\{M_t, m_t, t \geq 0\}$ where $M_{t+1} = E(M_t, m_t)$ and $m_{t+1} \in S(G, E(M_t, m_t), m_t)$.

The economic system can now be summarized by the list $\mathcal{E} = (e, G, D)$, and the equations of motion for the system are given by the response rules $\gamma(\mathcal{E}) = (E, S)$.

2.4 Equilibrium, Optimality and Incentive Compatibility

A non-cooperative equilibrium of an economic system $\mathcal{E}$ is a feasible position of the system given by a (3 N + 1)-tuple $(x^*, y^*, M^*, m^*) \in \mathbb{R}^{N+1} \times \mathcal{M} \times M$ that is unchanged by the equations of motions $\gamma(\mathcal{E})$. An equilibrium occurs when no consumer can improve his own situation (in terms of his response rule) by a unilateral action that is permissible under the given enforcement rule. In other words, m^* is a best replay or Nash equilibrium with respect to the response rules $S(G, M^*, \cdot)$ and the message space M^*. This does not mean that m^* is a Nash equilibrium relative to the base message space M, nor does it necessarily follow that M^* is a degenerate set. Formally, we say that,

Definition 3. $(x^*, y^*, M^*, m^*) \in W \times \mathcal{M} \times M$ is a non-cooperative equilibrium for the economic system $\mathcal{E}$ if

(a) $M^* = E(M^*, m^*)$;

(b) for each $i \leq N$, $m_i^* \in S^i(G, M^*, m^*)$;

(c) for each $i \leq N$, $F(m^*/m_i) = F(m^*)$ and $C^i(m^*/m_i) = C^i(m^*)$ for all $m_i \in S^i(G, M^*, m^*)$ where (m^*/m_i) is the vector m^* with ith component m_i, i.e. $m^*/m_i = (m_1^*, \ldots, m_{i-1}^*, m_i, m_{i+1}^*, \ldots, m_N^*)$;

(d) $(x^*, y^*) = (w - C(m^*), F(m^*))$.

We say that (M^*, m^*) is an E-Nash equilibrium for the system, thus indicating the possible dependence of the Nash equilibrium nature of m^* on the enforcement rule E.[9] Observe that if E is an evolving enforcement rule, then M^* reflects the complete adjustment history since $M^* = \lim_t E(M_t, m_t) = \lim_t M_{t+1}$ is then well-defined.[10]

Social performance of the process is measured by Pareto optimality as conventionally defined. Given an economy e, denote the set of feasible allocations (x, y) that satisfy the first order conditions for Pareto optimality by $\theta(e)$, called the (first order) Pareto set for e. For e as defined in section 2.1 above, $(x, y) \in \theta(e)$ if and only if

(a) for each $i \leq N$, $\lambda_i \frac{\partial u^i}{\partial x_i}(x_i, y) = 1$,

(b) $\sum_{i \leq N} \lambda_i \frac{\partial u^i}{\partial y}(x_i, y) = 1$

(c) $(x, y) \in W$,

where the λ_i, $i \leq N$ are utility weights, not all zero.[11]

In choosing a definition of individual incentive compatibility, our motive is to emphasize the incentive characteristics for agents' responses in equilibrium instead of their incentives while adjustments are occurring. Thus, individual incentive compatibility of an allocation mechanism G for an economy e is defined at equilibria of the economic system . However the formulation is in terms of a class of consumer response rules that satisfy a rational behavior hypothesis and feasibility constraints. The formulation is technically different from that of Hurwicz (1972) and Ledyard (1974), but is conceptually similar in that a wide class of non-competitive response rules is allowed.

Definition 4. Let $m_{-it} = (m_{1t}, \ldots, m_{(i-1)t}, m_{(i+1)t}, \ldots, m_{Nt})$ and $M_{-it} = \bigtimes_{\substack{j \leq N \\ j \neq i}} M_{jt}$ for each $t \geq 1$. For each $m_{-it} \in M_{-it}$, let $C^i(m_{-it}) = \{m_{it} \in M_{it} : C^i(m_{-it}, m_{it}) \leq w_i\}$, and let $\mathcal{S} = \bigtimes_{i \leq N} \mathcal{S}^i$ where, for each $i \leq N$, $\mathcal{S}^i$ is the class of response rules $S^i(\cdot)$ such that

(a) $m_{it} \in S^i(G, M_t, m_{t-1})$ implies $m_{it} \in M_{it}$, and

(b) for each $m_{it} \in S^i(G, M_t, m_{t-1})$, there exists $m_{-it} \in M_{-it}$ such that

  (i) $m_{it} \in C^i(m_{-it})$,

  (ii) $u^i(x_i, F(m_{-it}, m_{it})) \geq u^i(\hat{x}_i, F(m_{-it}, \hat{m}_{it}))$ for all $\hat{m}_{it} \in C^i(m_{-it})$ where $x_i = w_i - C^i(m_{-it}, m_{it})$.

A mechanism G is individually incentive compatible in e if, for all $S \in \mathcal{S}$, the response rules $\gamma(E) = (E, S)$ generate an equilibrium at which each consumer i is communicating a local representation of his true preferences.

To interpret Definition 4, recall that generally an allocation mechanism is said to be individually incentive compatible if no agent can improve his situation by following non-competitive behavior. Both Hurwicz and Ledyard interpret this to mean that no consumer should be able to obtain a preferred allocation (relative to his true preferences) by behaving as if his preferences are different from what they are while other consumers behave according to their true preferences. Definition 4 does not say this! Definition 4 permits each consumer to communicate essentially any preference strategy he

desires at each time t provided it is legally permissible (property (a)), is financially feasible for some permissible response of the other agents (property (b)(i)), and is designed to maximize his true preferences for some permissible response of the other agents (property (b)(ii)). This last requirement merely asserts that no agent should knowingly select a strategy that can only lead to an allocation that is less preferred than the minimum he can achieve regardless of the strategies selected by the other agents.

It would seem then that this is a straightforward generalization of the Hurwicz and Ledyard static formulations. This is not quite true because our final criterion is only that each agent be communicating local truth in equilibrium; this does not mean that some agent's equilibrium allocation cannot be improved as a consequence of communicating untruthful information during the adjustment process. To strengthen the definition to exclude this possibility would make an investigation of the incentive compatibility of G very difficult because of the nature of the adjustment process.

3. A CONVERGENT DEMAND-REVEALING ECONOMIC SYSTEM

In this section, we present an economic system that always converges to an allocation in $\theta(e)$, characterized by a balanced government budget and no consumer bankruptcy. Consumers' response rules are defined by a simple Cournot behavioral hypothesis, and the government's economic plan requires that consumers report willingness to pay schedules for the public good. The allocation and tax rules are similar to rules that have been considered in a static framework by Groves-Ledyard (1977), Groves-Loeb (1975) and Suchanek (1977). The enforcement rule imposes an intertemporal consistency on the willingness to pay schedules reported by consumers, given the allocation and tax rules.

The allocation rule is designed to select a level of public good production that maximizes reported consumer surplus, and the tax rules assess each consumer the difference between the cost of the public good and the aggregate willingness to pay for the public good reported by the other consumers plus a lump sum balancing payment (which may be negative). Thus, for each vector of agents' messages, these rules compute a level of public good production <u>and</u> a corresponding vector of individual tax burdens. Implicit in the message transmitted by each consumer is a schedule of reported marginal valuations for the public good. The enforcement rule binds each consumer to the tax burden - public good level correspondence computed at each iteration for all future iterations by restricting the subsequent sets of permissible messages.

Therefore, if a consumer communicates a strategic message at some point in time that misrepresents his true willingness to pay for the public good level computed, then he must continue to transmit messages that communicate this misrepresentation, <u>even if the other consumers react in such a way that</u>

<u>he is made worse off in the future</u>. Consumers may, however, change their reported marginal valuations for the public good within certain limits that become more restrictive over time. The principal theorems in this section state that all consumers communicate marginal truth in the limit, under various behavioral assumptions, from which it follows that the limiting allocation is Pareto efficient. Throughout the following analysis, let the economy e be as defined in Section 2.1.

3.1 An Economic Plan

Let $C^2(\mathbb{R},\mathbb{R})$ denote the set of all twice differentiable functions from $\mathbb{R}$ into $\mathbb{R}$, and let t denote time.

<u>Definition 5</u>. Let $\mathcal{M}$ be the set of all subsets of M, and let K_1, K_2, K_3, be positive constants. Define $G^* = \{M, F, (C^i, i \leq N), E\}$ by

(a) $M = \underset{i \leq N}{X} M_i$ where $M_i = \{m_i \in C^2(\mathbb{R},\mathbb{R})\colon\ m_i(0) = 0;\ 1 < m_i'(0) < K_1;$ $0 < m_i'(y)$ and $-K_2 \leq m_i''(y) \leq -K_3 < 0$ for all $y \geq 0\}$ for all $i \leq N$;

(b) for each $t \geq 0$ and $m_t \in M$, $F(m_t) = y_t > 0$ where y_t maximizes $\left[\sum_{i \leq N} m_{it}(y) - y\right]$,

(c) for each $t \geq 1$, $(m_s \in M, s \leq t)$ and $i \leq N$, $C^i(m_s, s \leq t) =$

$$F(m_t) - \sum_{\substack{j \neq i \\ j \leq N}} m_{jt}(F(m_t)) + \Pi_i(m_s, s \leq t-1)$$

$$= y_t - \sum_{j \neq i} m_{jt}(y_t) + \Pi_i(m_s, s \leq t-1);$$

(d) for each $t \geq 1$, $M_{t+1} = E(M_t, m_t) = \underset{i \leq N}{X} E_i(M_t, m_t) = \underset{i \leq N}{X} M_{it+1}$

where, for all $i \leq N$,

(i) $M_{i1} = M_i$,

(ii) $M_{i2} = \{m_i \in M_{i1}: \ m_{i2}(y_1) = m_{i1}(y_1) \text{ and } m_{i2}(y_I) \leq m_{i1}(y_I)\}$

where $y_I = \inf\{y \in \mathbb{R}_+: \ \sum_{i \leq N} m_{i1}(y) \leq y\}$,

(iii) $M_{it+1} = \{m_i \in M_{it} : \ m_{it+1}(y_t) = m_{it}(y_t)\}$ if $t \geq 2$.

Under the economic plan G^*, the base message space M_i for each consumer i is the set of all twice differentiable, strictly concave real-valued functionals with uniformly bounded first and second derivatives. At each time t, the allocation rule selects the public good level y_t that maximizes the difference between consumers' aggregate reported willingness to pay for the public good and the production costs (i.e., consumers' surplus). Existence of the y_t for each t is ensured by the uniform bounds on the derivatives of the permissible messages, making both the allocation rule and the tax rules well-defined, provided the lump sum components Π_i are well-defined.

The purposes of the Π_i are to avoid consumer bankruptcies and to ballance the government's budget in equilibrium, thereby ensuring feasibility of the limiting allocation and the validity of Pareto efficiency as a measure of performance, while not influencing consumers' decisions in the margin. Two things are to be noted at this point. Firstly, the Π_i are (potentially) functions of the entire adjustment history, and secondly, the Π_i have not

been specified. The first observation permits a specification of the Π_i that we prefer for normative reasons, although it is more easily criticized than alternative formulations. Yet, these balance terms are critical in the operation of the dynamic process, and for justification of our results. Therefore, we have elected to defer consideration of the balance terms until later in this section, which also permits us to distinguish more easily the generic properties of G^* that derive from the enforcement rule E and the properties of the tax rules.

The enforcement rule $E(\cdot)$ defines an evolving enforcement structure which, for each sequence of play, yields a nested sequence of sets M_{it}, $t \geq 1$ for each $i \leq N$. Each successive element M_{it+1} in the sequence depends directly on the preceeding element M_{it}, the message selected by the consumer m_{it}, and the public good level y_t generated by the allocation rule $F(\cdot)$ given the messages selected by all agents m_t. For each i, M_{i1} is the base message space; M_{i2} is the subset of M_{i1} that consists of the functions that preserve i's reported willingness to pay for y_1 as communicated by m_{i1}, his first message selection, and that do not communicate a willingness to pay for y_I that exceeds $m_{i1}(y_I)$. (Notice that y_I is such that $\sum_{i \leq N} m_{i1}(y_I), = y_I$, i.e., y_I is the public good level where reported consumers' surplus is zero at $t = 1$. The bounds on the derivatives ensure that y_I exists and is unique.) M_{i3} is the subset of M_{i2} that preserves i's reported willingness to pay for y_2 where y_2 maximizes $\left[\sum_{i \leq N} m_{i2}(y) - y\right]$, and so on for $t = 3, 4, \ldots$. Therefore, an intertemporal sequence of consistent message selections for each consumer i must be such that $m_{is}(y_t) = m_{it}(y_t)$ for all $s \geq t \geq 1$ and $m_{is}(y_I) \leq m_{i1}(y_I)$ for all $s \geq 1$. The purpose of y_I is technical: to limit consideration to public good levels belonging to a compact set that is endogenously determined by consumers themselves, given the production technology.

Lemma 1. Let $m_o \varepsilon M$ be an arbitrary initial condition, and let $\{M_t, m_t, t \geq 1\}$ be a message trajectory associated with m_o by an economic system $E = (e, G^*, D)$. Then E has the following properties independent of e and S (the economic environment and consumer response rules, respectively):

(i) there exists a unique set $M^* \varepsilon M$ such that $M_t \underset{t}{\rightarrow} M^*$;

(ii) there exists a unique 2-tuple $(y^*, k^*) \varepsilon \mathbb{R}^2_+$ such that $(y_t, [\sum_{i \leq N} m_{it}(y_t) - y_t]) \underset{t}{\rightarrow} (y^*, k^*)$

(iii) there exists a unique (N + 1)-tuple $(y^*; k_i^*, i \leq N)$ such that $(y_t; m_{it}(y_t), i \leq N) \underset{t}{\rightarrow} (y^*; k_i^*, i \leq N)$;

(iv) if the sequence $\{y_t\}$ contains a sub-sequence $\{y_{t_n}\}$ such that $y_{t_n} \neq y_{t_h}$ for $h \neq n$, then

(a) there exists a unique (N + 1)-tuple $(y^*; r_i^*, i \leq N)$ such that $(y_t; m'_{it}(y_t), i \leq N) \underset{t}{\rightarrow} (y^*; r_i^*, i \leq N)$, and

(b) for all $m^* \varepsilon M^*$, $m_i^*(y^*) = k_i^*$, $m_i^{*\prime}(y^*) = r_i^*$ and $\sum_{i \leq N} r_i^* = 1$;

(v) if, for each $i \leq N$, either $m'_{is}(y_t) \leq m'_{iv}(y_t)$ for all $s \geq v \geq t$ such that $y_t = y_s = y_v$, or $m'_{is}(y_t) \geq m'_{iv}(y_t)$ for all $s \geq v \geq t$ such that $y_t = y_s = y_v$, then there exists a unique (N+1)-tuple $(y^*; r_i^*, i \leq N)$ such that $(y_t; m'_{it}(y_t), i \leq N) \underset{t}{\rightarrow} (y^*; r_i^*, i \leq N)$.

Lemma 1 establishes the general convergence properties of the economic plan G^*. Intuitively, the enforcement rule $E(\cdot)$ is a procedure for eliciting information from consumers sufficient to construct individual willingness to pay schedules that permit the government to successively approximate a unique public good level y^* and vector of reported individual valuations $(k_i^*, i \leq N)$, using the allocation and tax rules. Whether a unique vector of individual marginal valuations of y^* is similarly approximated can depend on agents' response rules. Technically, this means that there exist response rules $S(\cdot)$ that will generate a periodicity asymptotically in the marginal valuations consistent with $(y^*; k_i^*, i \leq N)$. However, this cannot happen if $S(\cdot) \varepsilon S$ and if each $S^i(\cdot)$ satisfies a consistency axiom which effectively restricts consecutive directional changes in the reported marginal valuations for repeated tentative allocations of the public good. Thus, the same result can be achieved by the government if it adopts the following, more restrictive enforcement rule:

Definition 5(d'). for each i, $E_i(M_t, m_t) = M_{it+1} = \{m_i \varepsilon M_{it}: m_i(y_t) = m_{it}(y_t)$ and $m'_i(y_t) \geq m'_{it}(y_t)\}$.

The uniform boundedness imposed on the derivatives of allowable messages by the constants K_1, K_2, K_3 is the technical feature of the enforcement structure that literally forces convergence of the economic system, given, of course, the government's intertemporal reporting consistency requirement. To justify the use of these constants, we merely observe that they mean the government has assumed that individual preferences satisfy certain general properties, at least locally in a common region. K_1 means that no individual places an infinite value on an infinitesimal amount of the public good; K_2 and K_3 mean that the reported marginal rate of substitution in consumption of y for x cannot change arbitrarily quickly or too slowly, regardless of the relative consumptions of these commodities. In short, the reported intensity of preference of any consumer for the public good can never be too great or too small, nor can it change too drastically in a small region. Thus, these restrictions are simply regularity conditions on reported preferences. If agents behave as truthtellers, the restrictions will limit the set of potential equilibrium allocations to those that are in a region where consumers' marginal utility functions satisfy a uniform boundedness criterion corresponding to K_1. Notice, too, that this emphasizes that the government is attempting only to approximate the true state of affairs locally; the curvature restriction imposes no theoretical barrier to existence, nor does it imply that the government must have apriori knowledge of consumers' utility functions.

Lemma 1 shows that the vector of balance terms $\Pi = (\Pi_i, i \leq N)$ has no impact on the general convergence properties of the economic system. However, they are obviously critical for the questions of budget balance and consumer bankruptcy. Clearly, to balance the government's budget in equilibrium, these terms must be functions of the information communicated by consumers. This introduces the possibility of income effects that could affect the intended incentive structure of the economic plan if a consumer is sophisticated and can learn enough to manipulate the economic system. Generally, this problem

is assumed away in one form or another, often by making each Π_i a function only of the m_j, $j \neq i$, and then assuming that each consumer takes other consumers' messages as given (a Cournot competitive behavior assumption). This formulation is obviously not immune to the sophisticated consumer criticism.[13] Still, it is the approach we pursue here, but with a twist: the enforcement structure $E(\cdot)$ forces convergence of the economic system so that the Π_i too will have to converge if they are functions of the m_j only indirectly through the allocation rule. Moreover, this permits us to include in each Π_i the <u>ith</u> consumers <u>own</u> messages, although this is not necessary to obtain any principal result.[14] It does, however, yield an intuitively acceptable distribution of the cost of the public good.

<u>Definition 6</u>. For each $t \geq 1$, $\Pi_c(m_{t-1}) = (\Pi_{ic}(m_{t-1}), i \leq N)$ where

$$\Pi_{ic}(m_{t-1}) = \sum_{\substack{j \leq N \\ j \neq i}} m_{jt-1}(y_{t-1})\left[1 - \left(\frac{1}{\sum_{h \leq N} m_{ht-1}(y_{t-1})}\right)y_{t-1}\right].$$

<u>Theorem 1</u>. Let all starred values be as defined by lemma 1, and let $C^i(\cdot)$ be as given by definition 5(c) where $\Pi_i = \Pi_{ic}$ for each $i \leq N$. Then, for all $i \leq N$,

(i) $$\frac{\partial \Pi_{ic}(m_{t-1})}{\partial m_{it}} \equiv 0, \text{ all } t \geq 1;$$

(ii) $$\lim_{t \to \infty} C^i(m_t(y_i)) = \left(\frac{k_i^*}{\sum_{h \leq N} k_h^*}\right)y^*;$$

(iii) $$\lim_{t \to \infty} \sum_{i \leq N} C^i(m_t(y_t)) = y^*.$$

Proof. (i) follows immediately from definition 5(d) since $m_t(y_{t-1}) = m_{t-1}(y_{t-1})$ all t. To obtain (ii), just apply the limit operation to $C^i(\ )$ and appeal to lemma 1:

$$\lim_t C^i(m_t(y_t)) = \lim_t y_t - \sum_{j \neq i} \lim_t m_{jt}(y_t) +$$

$$\sum_{j \neq i} \lim_t m_{jt-1}(y_{t-1}) - \left[\frac{\sum_{j \neq i} \lim_t m_{jt-1}(y_{t-1})}{\sum_{h \leq N} \lim_t m_{ht-1}(y_{t-1})}\right] \lim_t y_{t-1}$$

$$= y^* - \left(\frac{\sum_{j \neq i} k_j^*}{\sum_{h \leq N} k_h^*}\right) y^*$$

$$= \left(\frac{k_i^*}{\sum_{h \leq N} k_h^*}\right) y^*.$$

(iii) follows immediately from (ii).//

Definition 6 states that each consumer i will receive a lump sum payment at t (provided contracting occurs) equal to the difference between other consumers' reported willingness to pay for the tentative public good level y_{t-1} and their percentage of the total reported willingness to pay the production cost of y_{t-1}. Two features of Π_c should be noted: at each t, it is a function only of m_{t-1} evaluated at y_{t-1}, values that are fixed by the government for all time, and Π_{ic} is a function of $m_{jt-1}(y_{t-1})$, all $j \leq N$. However, these features do not create obvious difficulties, regardless of agents' response rules, precisely because the values are fixed for all time by the government. Indeed, Theorem 1 shows that, (i) the Π_{ic} are incentive neutral at each iteration, a consequence of the enforcement structure, (ii) each consumer's percentage of the cost of providing the public good is asymptotically well-defined and equal to his proportion of society's aggregate reported willingness to pay for the public good, and (iii) society's tax burden in the limit is equal to the government's cost of providing the public good. Theorem 1 does not depend on the characteristics of agents' response rules.

Therefore, income effects and consumer overreactions cannot cause a violation of any general convergence property. Still, one might argue that Π_c is an undesirable specification of the balance terms because a sophisticated consumer would be able to easily manipulate the distribution of costs at any point in time. But, this is precisely what is not true. Consider the following specification of the balance terms:

Definition 6'. For each $t \geq 1$, $\Pi_a(m_s, s \leq t-1) = (\Pi_{ia}(m_s, s \leq t-1), i \leq N)$

where $\Pi_{ia}\ m_s, s \leq t-1) =$

$$\sum_{j \neq i} \sum_{s \leq t-1} \left(\frac{m_{js}(y_s)}{t-1}\right) \left[1 - \frac{\sum_{s \leq t-1}\left(\frac{y_s}{t-1}\right)}{\sum_{h \leq N} \sum_{s \leq t-1}\left(\frac{m_{hs}(y_s)}{t-1}\right)}\right].$$

No consumer can easily manipulate the term Π_{ia} at any given point in time since the impact of any one message on the value of Π_{ia} goes to zero as t approaches infinity. Yet, it is straightforward to show that theorem 1 is also valid for $\Pi = \Pi_a$, a consequence of the following well-known result:

Lemma 2. If a sequence of numbers a_n converges to a, then the sequence of arithemetic means $\left\{\sum_{n=1}^{T} \frac{a_n}{T}\right\}$ converges to a. That is, $\lim a_n = a$ implies $\lim_{T\to\infty} \sum_{n=1}^{T} \frac{a_n}{T} = a$.

We conclude the enforcement rule $E(\cdot)$ circumvents the problem of income effects by eliminating asymptotically each agents' ability to manipulate his income level directly. However, potentially substantial own income variations can be forced by any agent initially during the adjustment process.

3.2 Consumer Behavior and Incentive Compatibility

Theorem 1 shows that budget balance is a generic equilibrium property of an economic system $\mathcal{E} = (e, G^*, D)$. In this section, we explore the equilibrium properties of $\mathcal{E}$ when certain restrictions are placed on consumers' response rules. Briefly, we show that if consumers are either Cournot players or, are generally, consistent utility maximizers, then $\mathcal{E}$ converges to a Pareto efficient E-Nash equilibrium allocation; we show also that G^* is incentive compatible.

Assumption 2 (Cournot Hypothesis). For each $t \geq 1$, $i \leq N$, $m_{it} \in S_c^i(G^*, M_t, m_{t-1}) = \{m_i \in M_{it}: u^i(x_i, F(m_{t-1}/m_i)) \geq u^i(\hat{x}_i, F(m_{t-1}/\hat{m}_i))$ all $\hat{m}_i \in M_{it}$ such that $0 \leq C^i(m_{t-1}/m_i) = w_i - x_i$, $0 \leq C^i(m_{t-1}/\hat{m}_i) = w_i - \hat{x}_i\}$.

$S_c^i(G^*,\cdot,\cdot)$ is called the Cournot Response Rule.

The Cournot response rule is a competitive response rule; each consumer accepts the previous communications of other agents as parametric to his behavior. To see that $S_c^i(G^*,\cdot,\cdot)$ is well-defined, for each $t \geq 1$, let $Y_{it} = \{y \in \mathbb{R}_+: y = F(m_{t-1}/m_{it})$, some $m_{it} \in M_{it}\}$. Then,

Lemma 3. For each $t \geq 1$, Y_{it} is non-empty and compact, all $i \leq N$.

Since the Lagrangean

$$L_t^i(x_i, y) = u^i(x_i, y) - \rho^i(y - \textstyle\sum_{j\neq i} m_{jt-1}(y) + \Pi_i(m_{t-1}) + x_i - w_i)$$

is continuous in y for each t where $x_i = w_i + z_i$ is a simple residual, it follows from lemma 3 that, for each t and each i, there is a feasible utility maximizing message. We now state two technical lemmas that are used to prove optimality of the Cournot economic system.

Definition 7. for each $i \leq N$, $t \geq 1$, and $m_{it} \in S_c^i(G^*, M_t, m_{t-1})$, define the 2-tuple $(\tilde{x}_{it}, \tilde{y}_{it})$ to be the values of $(x_i, y) \in \mathbb{R}_+^2$ that satisfy the following conditions:

(a) $\tilde{y}_{it}$ maximizes $\left[m_{it}(y) + \sum_{j\neq i} m_{jt-1}(y) - y\right]$,

i.e. $\tilde{y}_{it} = F(m_{t-1}/m_{it})$, and

(b) $\tilde{x}_{it} = w_i - C^i(m_{t-1}, /m_{it}) \geq 0$.

Lemma 4. Let $m_{ht} \in S_c^h(G^*, M_t, m_{t-1})$ all $h \leq N$, $t \geq 1$. If $y_{t-1} = y_t$ some

$t \geq 1$, then for each i,

(i) $m'_{it}(y_{t-1}) < m'_{it-1}(y_{t-1})$ implies $m'_{it+1}(y_{t-1}) \leq m'_{it}(y_{t-1})$, and

(ii) $m'_{it}(y_{t-1}) > m'_{it-1}(y_{t-1})$ implies $m'_{it+1}(y_{t-1}) \geq m'_{it}(y_{t-1})$.

Lemma 5. For each $i \leq N$, $\lim_t \tilde{y}_{it} = \lim_t y_t$.

Lemma 4 establishes a choice consistency property of Cournot behavior relative to the government plan G^*. Essentially, it says that if, at some time t, the ith consumer's response to the previous message selections reflects a desired decrease (respectively, increase) in the level of the public good to be provided, then so will his next response at (t+1) should other consumers react at time t so as to cancel the impetus for change communicated by him at t. Because the uniform strict concavity limitation on allowable messages restricts the extent to which this can happen, the existence of unique individual asymptotic marginal valuations follows from lemmata 1 and 5.

Lemma 5 establishes the asymptotic relationship between the Cournot responses of each consumer and the government's behavior as dictated by the economic plan G^*. The lemma asserts that the feasible public good level desired by each consumer who is a Cournot player will coincide in the limit with the public good level provided by the government in accordance with the economic plan G^*. The result is not obvious, and requires the uniform strict concavity of allowable messages; if linear messages are permitted, then lemma 5 is not true in general.[15]

Theorem 2. The neoclassical Cournot economic system $E_c = (e, G^*, D)$ where $S = S_c$ converges to an E-Nash equilibrium (M^*, m^*) such that the corresponding equilibrium allocation (x^*, y^*) is Pareto efficient and no consumer is bankrupt; i.e. $(x^*, y^*) \in \theta(e)$ and $w_i \geq C^i(m^*) \geq 0$ all $i \leq N$.

Proof. The proof of theorem 2 essentially consists of showing that (x^*, y^*, M^*, m^*) satisfies definition 3, and that G^* is incentive compatible in equilibrium in the restricted, static sense of Hurwicz and Ledyard,[16] from which it follows that $(x^*, y^*) \in \theta(e)$.[17]

Let $M^* = \lim_t M_t$ which exists by lemma 1. By construction, $M^* = \underset{i\,N}{X} M_i^*$ where $M_i^* = \{m_i \in M_i : m_i \in \bigcap_{t\geq 1} M_{it}\}$. To show that $M^* = E(M^*, m^*)$—definition 3(a)— it suffices to show that

(2) $\quad S(G^*, M^*, m) = M^*$, all $m \in M^*$.

Suppose not. Then, for some $i \leq N$, $m \in M^*$, there exists $\hat{m}_i \in M_i^*$ such that $u^i(\hat{x}_i, F(m/\hat{m}_i)) > u^i(x_i, F(m))$ and $C^i(m/\hat{m}_i) = w_i - \hat{x}_i$. But, this means that $F(m/\hat{m}_i) \neq F(m)$ for otherwise, $C^i(m/\hat{m}_i) = C^i(m)$ and $\hat{x}_i = x_i$, a contradiction. It follows that there exists $\varepsilon > 0$ and $T > 0$ such that $|F(m_t) - F(m_t/m_{it+1})| > \varepsilon$ for all $t \geq T$, a contradiction of lemma 5. This establishes (2).

Observe that definition 3(b) and 3(c) also follow as consequences of (2). By lemmata 1 and 5, these conditions are all non-vacuous.

Now, by the Cournot hypothesis, $w_i \geq C^i(m_{t-1}/m_{it}) \geq 0$ for $m_{it} \in S^i(G^*, M_t, m_{t-1})$, all $t \geq 1$ and $i \leq N$. Thus, $w_i \geq \lim_t C^i(m_{t-1}/m_{it}) \geq 0$, and hence $w_i \geq \lim_t C^i(m_t) \geq 0$ by lemmata 1 and 5. Because of theorem 1, we can write this more simply as $w_i \geq \left(\frac{{}_i k^*}{\sum_{h \leq N} k_h^*}\right) y^* \geq 0$ where, by (2), $k^* = m_i(y_i^*)$, all $m_i \in M_i^*$, all $i \leq N$. Hence, $w_i \geq C^i(m_i^*) \geq 0$ all i, showing that no consumer is bankrupt. Since, further, $x_i^* = w_i - C^i(m_i^*)$ by construction for all i, definition 3(d) is established.

Thus, (M^*, m^*) is an E-Nash equilibrium.

It remains for us to show that $(x^*, y^*) \in \theta(e)$. Under the Cournot hypothesis, for each t, each consumer i selects a message m_{it}—which here determines x_{it} as a residual—to maximize the Lagrangean

$$L_t^i\,(x_i, m_i) = u^i(x_i, F(m_{t-1}/m_i)) \;- \rho_i(x_i + C^i(m_{t-1}/m_i) - w_i).$$

That is, i selects $(\tilde{x}_{it}, m_{it}) \in \mathbb{R}_+ \times M_{it}$ such that

(3) (a) $u^i_{x_i}\,(\tilde{x}_{it}, F(m_{t-1}/m_{it})) = \rho_i$,

(b) $u^i_{m_i}\,(\tilde{x}_{it}, F(m_{t-1}/m_{it})) = \rho_i\, C^i_{m_i}\,(m_{t-1}/m_{it})$.

This choice problem is well-defined by lemma 3.

Since $C^i_{m_i}\,(m_{t-1}/m_{it}) = 1 - \sum_{j \neq i} \frac{dm_{jt-1}}{dy}(\tilde{y}_{it}) = \frac{dm_{it}}{dy}(\tilde{y}_{it})$ under the Cournot hypothesis and G^*, it follows from lemmata 1, 4 and 5 that

(4) (a) $u^i_{x_i}\,(x_i^*, F(m^*)) = \rho_i$,

(b) $u^i_{m_i}\,(x_i^*, F(m^*)) = \rho_i\, r_i^*$,

where $F(m^*) = y^*$ and $r_i^* = \frac{dm_i^*}{dy}(y^*) = \lim_t \frac{dm_{it}}{dy}(y_t)$ for all $i \leq N$. Clearly, $\sum_{i \leq N} r_i^* = 1$, a simple limit result. Letting $\lambda_i = \rho_i^{-1}$ for each $i \leq N$ yields conditions (1)(a) - (d). Thus, $(\bar{x}^*, y^*) \in \theta(e)$.//

Observe that 4(b) means that each consumer is implicitly communicating in the limit his true marginal willingness to pay for the equilibrium public good level y^*. In short, when viewed statically, the mechanism G^* is individually incentive compatible in the sense of Hurwicz (1972) and Ledyard (1974). This follows since the Cournot hypothesis in equilibrium is indistinguishable from the static competitive behavior assumption of Hurwicz and Ledyard, and because a consumer here could act over time as if he is communicating demand information consistent with virtually any concave utility function, not just his true utility function.

A stronger incentive compatibility-efficiency result is possible. Theorem 3, below, establishes that G^* is individually incentive compatible in the sense of definition 4, provided consumers' behaviors satisfy a choice consistency or monotonicity property. This is equivalent to showing that G* is an efficient economic plan that does not induce waste or bankruptcy for a large class of consumer response rules.

Implicit in the ith consumer's Cournot response rule is the expectation by i at each time t that the response of other consumers at (t-1) will be recommunicated. Similarly, some notion of expectation must be implicit in each response rule belonging to the classes S^i, $i \leq N$. For each i and each S^i, there is a particular expectation formulation that associates with each set of possible responses for other consumers and the previous communications a subset consisting of an equivalence class of their expected responses: $\eta_i: M_{-i} \times M \to M_{-i}$ where $\eta_i(M_{-it}, m_{t-1}) = M^S_{-it} \subset M_{-it}$. That is, a specific expectation correspondence η_i and utility maximizing behavior together fully determine a unique $S^i \in S^i$. We focus in theorem 3 below on the sub-class of response rules which, with their associated expectation correspondence, satisfy a consistency property that ensures convergence of the individual marginal valuations, or equivalently rules out cyclical communication behavior.

Definition 8. $S^i \in S^i$ satisfies the axiom of consistency when

(a) $M^S_{-it} \subset M_{-it}$ for all $t \geq 1$, and

(b) if $y_{t-1} = y_t$ for some $t \geq 1$, then $m^S_{-it+1} \in M^S_{-it+1}$ implies

(i) $$\sum_{j\neq i} \frac{dm^S_{jt+1}}{dy}(y_t) \geq \sum_{j\neq i} \frac{dm_{jt}}{dy}(y_t) \text{ if}$$

$$\sum_{j\neq i} \frac{dm_{jt}}{dy}(y_t) \geq \sum_{j\neq i} \frac{dm_{jt-1}}{dy}(y_{t-1}), \text{ and}$$

$$\sum_{j\neq i} \frac{dm^S_{jt+1}}{dy}(y_t) \leq \sum_{j\neq i} \frac{dm_{jt}}{dy}(y_t) \text{ if}$$

$$\sum_{j\neq i} \frac{dm_{jt}}{dy}(y_t) \leq \sum_{j\neq i} \frac{dm_{jt-1}}{dy}(y_{t-1}), \text{ or}$$

(ii) $$\frac{\partial \eta_i (M_{-it}, m)}{\partial m} \equiv 0 \text{ all } t \geq 1.$$

The axiom of consistency is a regularity condition on the expectations correspondence $\eta_i(\cdot)$ implicit in $S^i(\cdot)$. Essentially, the axiom requires that each consumer's expectations be feasible,and that it exhibit Cournot-like behavior when consecutive communications result in the same tentative allocation being computed by the government. More specifically, it requires that expected communications reflect an expectation that the aggregate marginal valuations reported by other consumers will change in the same direction as occurred previously if the same tentative allocation is computed consecutively by the government. Moreover, it requires that a consumer's responses not change if the expected responses of others have not changed. Under the axiom, it is possible to show that individual decisions and the government's decision conform in the limit, thus ruling out cyclical strategy behavior that leaves the real position of the economy unchanged.

Lemma 6. If $S^i \varepsilon S^i$ satifies the axiom of consistency for all $i \leq N$, then

(i) $\lim_t m_{it}(y_t)$ exists, and

(ii) $\lim_t \tilde{y}_{it} = \lim_t y_t$

where for each $t \geq 1$, $\tilde{y}_{it}$ is the value of y generated by S^i in accordance with definition 7, and y_t is the value of y generated by G^*.

Theorem 3. The neoclassical economic system $E = (e, G^*, D)$, when S^i satifies the axiom of consistency for all i, converges to an E-Nash equilibrium (M^*, m^*) such that the corresponding equilibrium allocation $(x^*, y^*) \varepsilon \theta(e)$ and $w_i \geq C^i(m^*) \geq 0$ all $i \leq N$.

The proof of theorem 3 parallels that of theorem 2, following from the application of lemmata 1 and 6; one need only substitute $m^s_{-it} \varepsilon \eta_i(M_{-it}, m_{t-1})$ for m_{t-1} in the proof to theorem 2. Observe that the Cournot response rule

satisifies the axiom of consistency, making theorem 2 a corrollary of theorem 3. If the government adopts the more restrictive enforcement rule $E(\cdot)$ given by definition 5(d'), then the following optimality theorem can be proven, again by paralleling the proof of theorem 2:

Theorem 4. The neoclassical economic system $E = (e, G', D)$, where G' is defined by definition 5(a) - (c) and 5(d'), and where $S \varepsilon S$, converges to an E-Nash equilibrium (M^*, m^*) such that the corresponding equilibrium allocation $(x^*, y^*) \varepsilon \theta(e)$ and $w_i \geq C^i(m^*) \geq 0$ all $i \leq N$.

Theorem 4 establishes that there is an institutional design that forces convergence to an efficient equilibrium allocation if consumers pursue a utility maximizing strategy regardless of the otherwise manipulative nature of their response rule.

Theorems 2, 3 and 4 are valid independent of the initial condition $m_o \varepsilon M$ communicated by the government to begin the adjustment process, although the equilibrium allocation may clearly vary with the initial condition. Therefore, it is of interest to know whether the economic system is stable in the sense that two initial conditions that are close will result in equilibria that are also close. Unfortunately, it appears that this is not the case. The reason for this is technical, and we discuss it further in the next section.

Finally, it may seem at first glance that the income effects issue has been finessed since direct manipulation of the balance terms Π_i, $i \leq N$ has not been modelled as part of the agents' messages choice problems. If true, this would raise the important question of whether the critical lemmata, 5 and 6, are valid when direct own-income manipulations are permitted. However, it is straightforward to show that the lemmata are still valid since the proofs are independent of agents' choice criteria; the proofs depend only on the outcome rule and each agent's expectations of other agents' responses which are binding constraints at each time t regardless of the agent's choice criteria.

4. CONCLUSION

Much of the recent research on incentive compatibility, Pareto efficient allocation rules, and the free rider problem stem from the seminal paper by Groves-Ledyard (1977), and earlier, related papers by Groves (1973) and Groves-Loeb (1975). All of these efforts present a similarly structured institution designed to resolve efficiently different types of communal decision problems. One can in fact view the entire comparative statics demand revelation literature as a collection of exercises in optimal institutional design. This is not true of the limited work on the implementability problem which has sought primarily to resolve differences between the existing theory and the experimental convergence results of Smith (1979) and others. The purpose of this paper is different: to develop an institutional structure that always leads to an efficient equilibrium allocation of resources, circumventing the barriers to theoretical convergence that plague less restrictive mechanisms, and to explore the relationship between agents' response rules and the institutional design given convergence and efficiency requirements. Thus, we reemphasize the economic policy nature of probable solutions to the basic efficiency problem confronting governments when viewed in a dynamic framework.

The economic policy approach to the implementability issue is pragmatic, and permits mechanisms such as G^* relative to which the economic system always converges to an efficient equilibrium for a wide class of economies without waste or bankruptcy of any agent in the process. Moreover, income effects need not be assumed away, but can be eliminated asymptotically by appropriate specification of the government's economic plan. How restrictive the government's enforcement rule must be to achieve these results depends on agents' response rules. But, theorem 4 shows that there does exist a

mechanism that will work for all utility maximizing response rules and a wide class of economies.

Our results and the model are not immune to criticism. From a practical point of view, the procedure is tâtonnement, and we have increased the informational load on the government of an already information intensive system. We do not think, however, that these shortcomings void the potential usefulness of the approach when applied to a small number of easily policed agents.

From a theoretical point of view, the Model is naive and cumbersome, possessing no meaningful theory of production, having only two goods, ignoring price adjustment problems in private good markets, and requiring consumers to communicate entire functions. To generalize the Model to eliminate any of these criticisms is, we believe, technically non-trivial.

From a policy point of view, the mechanism G^* has two undesirable characteristics. Although the mechanism is incentive compatible in equilibrium, the adjustment process is not necessarily incentive compatible. Some agent may be able to manipulate the iterative procedure to affect the equilibrium outcome significantly. On the other hand, Thomson's (1979) results are very suggestive, and it seems plausible that other agents may be able to counter-manipulate, essentially negating each individual agent's attempts at manipulative control. How such a manipulative process might be affected by the increasing restrictions imposed on the sets of allowable messages over time by the government remains to be explored, although it seems clear that an agent's ability to manipulate the process varies inversely with the rate of convergence of the system.

Finally, the economic system does not necessarily move from close initial conditions to close equilibrium allocations. Hence, the government cannot manipulate the equilibrium distribution of income by varying the initial condition, depriving it of an obvious potential policy instrument. This problem arises because each agent's set of possible (equivalent) responses at any finite time is, loosely speaking, too large. For example, at each time t, a Cournot player i, can choose from an infinite number of time-specific equivalent responses designed to yield a desired allocation y_{it} given the previous communications of other agents at time (t-1). However, most of these responses will result in different tentative allocations y_t computed by the government given the actual responses of other agents at time t. Because i's future messages are restricted by each tentative allocation y_t and his choice of message m_{it}, widely divergent paths can emanate from the same initial condition on two plays of the same tâtonnement game; messages that seemed equivalent at time t to a player i viewing the history need not be equivalent to the same player at time (t+s) viewing the new history. Whether the process would tend to stabilize with multiple plays of the same game given the same initial condition (a supergame) remains to be determined.

APPENDIX

A.1. Notation

To conserve space, we adopt the following notation for presentation of the proofs to the technical lemmas stated in the text:

$\exists$ ≡ there exists	$m_t(y) = \sum_{i\leq N} m_{it}(y)$
$\ni$ ≡ such that	$m'_t(y) = \sum_{i\leq N} m'_{it}(y)$
$\Longrightarrow$ ≡ implies	
$!$ ≡ unique	$m''_t(y) = \sum_{i\leq N} m''_{it}(y)$
$\therefore$ ≡ therefore	
$\forall$ ≡ for each	$m_{-it}(y) = \sum_{j\neq i} m_{it}(y)$
$\to$ ≡ converges to	
	$m'_{-it}(y) = \sum_{j\neq i} m'_{it}(y)$

A.2. Proofs

<u>Lemma 1</u>. (i) follows since $M_{t+1} \subset M_t$ all t. Now observe that $\{(y_t, m_t(y_t)), t\geq 0\}$ is a bounded sequence. Hence, $\exists$ a convergent subsequence $\{(y_s, m_s(y_s))\}$. Let $(y^*,h^*) = \lim_s (y_s, m_s(y_s))$. To prove (ii), we must show that all subsequences converge to (y^*,h^*).

Suppose $\exists$ a subsequence $\{(y_\tau, m_\tau(y_\tau))\}$ with limit $(\bar{y},\bar{h}) \neq (y^*,h^*)$. Define $d_t = (m_t(y_t) - y_t)$ all t. By definition 5, $\{d_t, t\geq 1\}$ is bounded, monotone non-decreasing. $\therefore \exists ! \, d<\infty \ni d_t \to d \Longrightarrow (\bar{h} - \bar{y}) = (h^* - y^*)$. without loss of generality, assume $\bar{h} > h^*$. Then,

(5) $(\bar{h} - h^*) = (\bar{y} - y^*) > 0.$

Let $\varepsilon > 0$. $\exists$ integers $J_1(\varepsilon)$, $J_2(\varepsilon) < \infty \ni$

$$s \geq J_1(\varepsilon) \Longrightarrow \begin{cases} | m_s(y_s) - h^* | < \varepsilon/4, \\ | y_s - y^* | < \varepsilon/4. \end{cases}$$

$$\tau \geq J_2(\varepsilon) \Longrightarrow \begin{cases} |\overline{h} - m_\tau(y_\tau)| < \varepsilon/4, \\ |\overline{y} - y_\tau| < \varepsilon/4. \end{cases}$$

Thus,

(6) (a) $|m_\tau(y_\tau) - m_s(y_s)| > |\overline{h} - h^*| - \varepsilon/2$

(b) $|y_\tau - y_s| > |\overline{y} - y^*| - \varepsilon/2$, and $|\overline{y} - y^*| > |y_\tau - y_s| - \varepsilon/2$.

for $s \geq J_1(\varepsilon)$, $\tau \geq J_2(\varepsilon)$. From (5), (6) (a), (b), we have,

(7) $|m_\tau(y_\tau) - m_s(y_s)| > |y_\tau - y_s| - \varepsilon$.

Let $\tau \geq J_2(\varepsilon)$ and $s \geq \max(\tau, J_1(\varepsilon))$. Then, $m_s(y_\tau) = m_\tau(y_\tau)$ and $m_t'(y_t) = 1$ all t by construction of G^*. $\therefore$, by Taylor's Theorem,

(8) $$m_s(y_\tau) - m_s(y_s) = (y_\tau - y_s)\, m_s'(y_s) + \int_{y_s}^{y_\tau} (y_\tau - y)\, m_s''(y)dy \leq (y_\tau - y_s)\,(1 - K_3/2)$$

since $m_s''(y) \leq -K_3 < 0$ all $y \geq 0$ by definition. It follows from (5), (6)(a),(b), (8) that

(9) (a) $m_\tau(y_\tau) - m_s(y_s) \leq (y_\tau - y_s)\,(1 - K_3/2)$,

(b) $m_\tau(y_\tau) - m_s(y_s) > |\overline{y} - y^*| - \varepsilon/2$.

Now, let $0 < \varepsilon < 2K_3\, |\overline{y} - y^*|/(4 + K_3)$. Then, $2K_3\, |\overline{y} - y^*|/(4 + K_3)$ $= \frac{1}{2} K_3\, |\overline{y} - y^*|/(1 + K_3/4)$, and thus $\varepsilon < \frac{1}{2} K_3\, (|\overline{y} - y^*| - \varepsilon/2)$ $< \frac{1}{2} K_3 |y_\tau - y_s|$ by (6)(b). $\therefore$ $|y_\tau - y_s| - \varepsilon > |y_\tau - y_s|\,(1 - \frac{1}{2}K_3)$. But, $|\overline{y} - y^*| - \varepsilon/2 > |y_\tau - y_s| - \varepsilon$ by (6)(b). Hence, $|\overline{y} - y^*| - \varepsilon/2 >$ $|y_\tau - y_s|(1 - \frac{1}{2} K_3)$. Thus, $m_\tau(y_\tau) - m_s(y_s) > m_\tau(y_\tau) - m_s(y_s)$ by (9)(a), (b), a contradiction.

The proof of (iii) is similar, following from the observations that $y_t \to y^*$ and that $|m_{i\tau}(y_\tau) - m_{is}(y_s)| = |m_{i\tau}(y_\tau) - m_{i\tau}(y_s)|$

$$\leq |y_\tau - y_s| \frac{(2K_1 + K_3)}{2},$$

a consequence of Taylor's Theorem and the definition of G^* for $\tau \geq s$.

To prove (iv), suppose $\exists$ subsequences $\{(y_\tau, m_{i\tau}(y_\tau))\}$, $\{(y_s, m_{is}(y_s))\}$ for some $i \leq N$ $m_{i\tau}'(y_\tau) \to r_i$ and $m_{is}'(y_s) \to r_i^* \neq r_i$ where $y_\tau, y_s \to y^*$ by (ii), above. By hypothesis, we may assume without loss of generality that $y_{\tau_\alpha} \neq y_{\tau_\beta}$ and $y_{s_\sigma} \neq y_{s_\lambda}$ all $\tau_\alpha \neq \tau_\beta$, $s_\sigma \neq s_\lambda$. Two cases are possible.

<u>Case I</u>. $\exists$ $\tau_\alpha, \tau_\beta, s_\lambda$, arbitrarily large $\ni$ $y_{\tau_\alpha} < y_{s_\lambda} < y_{\tau_\beta} \Longrightarrow m_{i\tau_\alpha}'(y_{\tau_\alpha}) > m_{is_\lambda}'(y_{s_\lambda}) > m_{i\tau_\beta}'(y_{\tau_\beta})$ since $m_{it}'(y) < 0$ all i, t. It follows that $r_i = r_i^*$.

<u>Case II</u>. $\exists$ $\tau_\alpha, \tau_\beta, s_\lambda, s_\sigma, s_\xi$ arbitrarily large $\ni$ $y_{\tau_\alpha} < y_{\tau_\beta} < y^* < y_{s_\xi} < y_{s_\lambda} < y_{s_\sigma}$ <u>and</u> $s_\xi > \tau_\beta > s_\lambda > \tau_\alpha > s_\sigma \Longrightarrow$ by definition of G^* that

$$\left[\frac{m_{i\tau_\beta}(y_{\tau_\beta}) - m_{i\tau_\alpha}(y_{\tau_\alpha})}{y_{\tau_\beta} - y_{\tau_\alpha}}\right] > m_{is_\xi}'(y^*) > \left[\frac{m_{is_\sigma}(y_{s_\sigma}) - m_{is_\lambda}(y_{s_\lambda})}{y_{s_\sigma} - y_{s_\lambda}}\right]$$

which can be rewritten as,

$$\left[\frac{m_{i\tau_\beta}(y_{\tau_\beta}) - m_{i\tau_\beta}(y_{\tau_\alpha})}{y_{\tau_\beta} - y_{\tau_\alpha}}\right] > m_{is_\xi}'(y^*) > \left[\frac{m_{i\tau_\beta}(y_{s_\sigma}) - m_{i\tau_\beta}(y_{s_\lambda})}{y_{s_\sigma} - y_{s_\lambda}}\right]$$

It follows that $r_i = r_i^*$.

To prove (v), it suffices to observe that either $\exists$ a subsequence

$\{y_\tau\} \ni y_{\tau_\alpha} \neq y_{\tau_\beta}$ if $\tau_\alpha \neq \tau_\beta$ in which case (iii) above applies, or $\exists\, J < \infty \ni y_t = y_s$ all t, $s \geq J$ in which case the sequence $\{m'_{it}(y_t),\ t \geq J\}$ is monotone by hypothesis, and hence converges.//

Lemma 2. Let $\varepsilon > 0$ be given. Observe that

$$\left|\frac{\sum_{n=1}^{T} a_n}{T} - a\right| = \left|\frac{a_1 + \ldots + a_T - Ta}{T}\right|$$

$$= \left|\frac{(a_1 - a) + \ldots + (a_{T_\varepsilon} - a)}{T}\right| + \left|\frac{(a_{T_\varepsilon+1} - a) + \ldots + (a_T - a)}{T}\right|$$

for $1 \leq T_\varepsilon \leq T$. By hypothesis, $\exists\, T_\varepsilon > 0 \ni$ $|a_n - a| < \varepsilon$ all $n \geq T_\varepsilon$. Let $T > T_\varepsilon$. Then,

$$\left|\frac{(a_{T_\varepsilon+1} - a) + \ldots + (a_T - a)}{T}\right| \leq \left(\frac{T-T_\varepsilon}{T}\right)\varepsilon, \text{ and } \left|\frac{(a_1 - a) + \ldots + (a_{T_\varepsilon} - a)}{T}\right| \leq$$

$\frac{T_\varepsilon}{T} \max_{n \leq T_\varepsilon} |a_n - a| \leq \frac{T_\varepsilon}{T}\varepsilon$. Hence, $\left|\frac{\sum_{n=1}^{T} a_n}{T} - a\right| \leq \left(\frac{T-T_\varepsilon}{T}\right)\varepsilon + \frac{T_\varepsilon}{T}\varepsilon = \varepsilon$. Since ε was arbitrary, the conclusion follows.//

Lemma 3. $Y_{it} \neq \emptyset$ since $y_{t-1} \in Y_{it}$ all $t \geq 1$. Since $Y_{it} \subset Y_{i1}$ all t, a bounded set, it remains only to show that Y_{it} is closed $\forall$ t. Let $\{y_{t_n}\} \subset Y_{it} \ni y_{t_n} \underset{n}{\to} \bar{y}$. We must show that $\bar{y} \in Y_{it}$, i.e. that $\exists\, m_{it} \in M_{it} \ni m'_{-it-1}(\bar{y}) + m'_{it}(\bar{y}) = 1$.

Define $r_{it}(y) = 1 - m'_{-it-1}(y)$. $r_i(\cdot)$ is continuous $\forall$ t; $\therefore$ $r_{it}(\bar{y})$ is well-defined, and $r_{it}(\bar{y}) = \lim_n r_{it}(y_{t_n})$. Hence, $r_{it}(\bar{y}) = \lim_n m'_{it_n}(y_{t_n})$ where, by hypothesis, $m_{it_n} \in M_{it}$, $m'_{it_n}(y_{t_n}) = r_{it}(y_{t_n})$. By construction of G^*, $\{m'_{it_n}(\cdot)\}$ is uniformly bounded. $\therefore$, for n=1, $\exists\, q_{i1} < \infty$ and a subsequence $\{m_{it_{n_1}}\} \subset \{m_{it_n}\} \ni m'_{it_{n_1}}(y_1) \underset{n_1}{\to} q_{i1}$, for n=2, $\exists\, q_{i2} < \infty$ and a subsequence $\{m_{it_{n_2}}\} \subset \{m_{it_{n_1}}\} \ni m'_{it_{n_2}}(y_2) \underset{n_2}{\to} q_{i2}$, etc., for n=t-1, $\exists\, q_{it-1} < \infty$ and a subsequence $\{m_{it_{n_{t-1}}}\} \subset \{m_{it_{n_{t-2}}}\} \ni m'_{it_{n_{t-1}}}(y_{t-1}) \underset{n_{t-1}}{\to} q_{it-1}$.

Now, consider the set of twice differentiable functions, $H_{it} = \{h_i \in C^2(\mathbb{R}, \mathbb{R}):\ h_i(0) = 0;\ h_i(y_s) = m_{is}(y_s)\ \forall\, s \leq t-1;\ h'_i(y_s) = q_{is}\ \forall\, s \leq t-1;\ h_i(\bar{y}) = r_{it}(\bar{y})\}$. Clearly, it suffices to show that,

Claim: $H_{it} \cap M_{it} \neq \emptyset$.

Suppose the claim is not true. Then $\exists\, m_i \in M_{it} \ni m'_i(y_s) \neq q_{is}$ some $s \leq t-1$, or $m'_i(\bar{y}) \neq r_{it}(\bar{y})$, or equivalently, $\forall\, h_i \in H_{it}$, either

(A1) $h''_i(y_s) < -K_2$ or (A2) $h''_i(y_s) > -K_3$, some $s \leq t-1$,

or

(B1) $h''_i(\bar{y}) < -K_2$ or (B2) $h''_i(\bar{y}) > -K_3$.

But, if (A1), then $\exists\, J < \infty \ni j \geq J \Longrightarrow m''_{it_{n_j}}(y_s) < -K_2$, and if (A2), then $\exists\, J < \infty \ni j \geq J \Longrightarrow m''_{it_{n_j}}(y_s) > -K_3$, which contradict the fact that $m_{it_n} \in M_{it}$, all n. Similar contradictions arise if (B1) or

(B2) are assumed. It follows that $H_{it} \cap M_{it} \neq \emptyset$, and the lemma is established.//

Lemma 4. To prove (i), it suffices to show that m_{it+1} is such that $\tilde{y}_{it+1} \leq y_{t-1} = y_t$, since $m'_{it}(y_{t-1}) < m'_{it-1}(y_{t-1}) \Rightarrow \tilde{y}_{it} < y_{t-1}$, and $m'_{it+1}(y_{t-1}) > m'_{it}(y_{t-1}) \Rightarrow \tilde{y}_{it+1} > y_{t-1}$.

Suppose $m'_{it+1}(y_{t-1}) > m'_{it}(y_{t-1})$. Then, $\tilde{y}_{it} < y_{t-1} = y_t < \tilde{y}_{it+1}$.

Observe that $\tilde{x}_{it} = w_i - \tilde{y}_{it} + m_{-it-1}(\tilde{y}_{it}) - \hat{\Pi}_i(y_{t-1})$, $x_{it-1} = w_i - y_{t-1} + m_{-it-1}(y_{t-1}) - \hat{\Pi}_i(y_{t-1}) = x_{it}$, and $\tilde{x}_{it+1} = w_i - \tilde{y}_{it+1} + m_{-it}(\tilde{y}_{it+1}) - \hat{\Pi}_i(y_{t-1})$ since $\hat{\Pi}_i(y_t) = \hat{\Pi}_i(y_{t-1})$, where $\hat{\Pi}_i(y_{t-1}) = \Pi_i(m_{-it-1})$ is given by definition 6.

Moreover,

$$(10) \quad \begin{cases} u^i(\tilde{x}_{it}, \tilde{y}_{it}) > u^i(x_{it-1}, y_{t-1}), \\ u^i(\tilde{x}_{it+1}, \tilde{y}_{it+1}) > u^i(x_{it-1}, y_{t-1}), \end{cases}$$

by definition 7.

Without loss of generality, assume that $u^i(\tilde{x}_{it}, \tilde{y}_{it}) > u^i(\tilde{x}_{it+1}, \tilde{y}_{it+1})$.

By the semi-strict quasi-concavity of $u^i(\cdot)$--assumption 1(ii)--we have,

$$(11) \quad u^i(\tilde{x}_i(\lambda), \tilde{y}_i(\lambda)) \geq u^i(\tilde{x}_{it+1}, \tilde{y}_{it+1}) \text{ all } \lambda \in (0,1)$$

where $\tilde{y}_i(\lambda) = \lambda\tilde{y}_{it} + (1-\lambda)\tilde{y}_{it+1}$ and $\tilde{x}_i(\lambda) = \lambda\tilde{x}_{it} + (1-\lambda)\tilde{x}_{it+1}$.

Since $\tilde{y}_{it+1} > y_{t-1} > \tilde{y}_{it}$, $\exists\ \lambda^* \in (0,1) \ni \tilde{y}_i(\lambda^*) = y_{t-1} \Rightarrow \tilde{x}_i(\lambda^*) = x_{it-1}$. $\therefore u^i(\tilde{x}_i(\lambda^*), \tilde{y}_i(\lambda^*)) = u^i(x_{it-1}, y_{t-1})$, a contradiction of (10) and (11).

A similar development proves (ii).//

Lemma 5. Recall that $\forall\ i \leq N$, all t,

$$(12) \quad m'_{it}(\tilde{y}_{it}) + m'_{-it-1}(\tilde{y}_{it}) = 1,$$

$$(13) \quad m'_{it}(y_t) + m'_{-it}(y_t) = 1,$$

$$(14) \quad m'_{it-1}(y_{t-1}) + m'_{-it-1}(y_{t-1}) = 1.$$

(12) and (13) $\Rightarrow$

$$m'_{it}(y_t) - m'_{it}(\tilde{y}_{it}) = m'_{-it-1}(\tilde{y}_{it}) - m'_{-it}(y_t)$$
$$= m'_{-it-1}(\tilde{y}_{it}) - m'_{-it-1}(y_{t-1})$$
$$+ m'_{-it-1}(y_{t-1}) - m'_{-it}(y_t)$$
$$\underset{t}{\to} m'_{-it-1}(\tilde{y}_{it}) - m'_{-it-1}(y_{t-1})$$

since $m'_{-it-1}(y_{t-1}) - m'_{it}(y_t) \underset{t}{\to} 0$ by lemmata 1, 4.

By the Mean Value Theorem, $\exists\ \xi_{it}$ and $\zeta_{it} \ni m'_{it}(y_t) - m'_{it}(\tilde{y}_{it}) = m''_{it}(\xi_{it})(y_t - \tilde{y}_{it})$, and $[m'_{-it-1}(\tilde{y}_{it}) - m'_{-it-1}(y_{t-1})] + [m'_{-it-1}(y_{t-1}) - m'_{-it}(y_t)] = m''_{-it-1}(\zeta_{it})(\tilde{y}_{it} - y_{t-1}) + o(1)$.

If $\tilde{y}_{it} \underset{t}{\to} y^* = \lim_t y_t$, then we're done; if not, then $\exists$ a subsequence

$\{\tilde{y}_{it}\} \ni \tilde{y}_{it} \to \bar{y} \neq y^*$. Now consider

$$(15) \qquad m''_{it}(\xi_{it}) \frac{(y_t - \tilde{y}_{it})}{|\bar{y} - y^*|} = m''_{-it-1}(\zeta_{it}) \frac{(\tilde{y}_{it} - y_{t-1})}{|\bar{y} - y^*|} + o(1).$$

For t large, $(y_t - \tilde{y}_{it}) \gtrless 0 \Leftrightarrow (\tilde{y}_{it} - y_{t-1}) \lessgtr 0$. Without loss of generality, assume $(y_t - \tilde{y}_{it}) > 0$ for t large. Then, $\frac{(y_t - \tilde{y}_{it})}{|\bar{y} - y^*|} \underset{t}{\to} 1$ and $\frac{(\tilde{y}_{it} - y_{t-1})}{|\bar{y} - y^*|} \underset{t}{\to} -1$. Hence, $\exists\ T < \infty \ni$ (15) $\Rightarrow$

$$(16) \qquad m''_{it}(\xi_{it}) = m''_{-it-1}(\zeta_{it}) + o(1) \text{ for } t \geq T.$$

But, by definition of G^*, $m''_{it}(\xi_{it}) \in [-K_2, -K_3]$ and $-m''_{-it-1}(\zeta_{it}) \in [K_3, K_2]$, a contradiction of (16) for t large.//

Lemma 6. There are two cases to consider: I. $\exists$ a subsequence $\{y_{t_n}\} \ni y_{t_n} \neq y_{t_h}$ all $n \neq h$, and II. $\exists\ T < \infty \ni y_t = y_s$ all $t,s \geq T$.

Case I. By lemma 1(iv), $\exists\ r_i^* < \infty \ni m'_{it}(y_t) \to r_i^*$, and $\sum_{i \leq N} r_i^* = 1$ which proves (i).

To prove (ii), suppose that $\tilde{y}_{it_n} \underset{n}{\to} \bar{y} \neq y^* = \lim_t y_t$ for some subsequence $\{\tilde{y}_{it_n}\} \ni \tilde{y}_{it_n} \neq \tilde{y}_{it_h}$ if $n \neq h$. Without loss of generality, assume $\bar{y} < y^*$.

$M^S_{-it} \subset M_{-it}$ all $t \Rightarrow \limsup_n M^S_{-it_n} \subset M^*_{-i} = \lim_t M_{-it}$. $\therefore 1 - \lim_n \left[\frac{dm^S_{-it_n}}{dy}(y^*)\right] = 1 - \lim_t m'_{-it}(y^*)$ since $m'_{-i}(y^*) = \hat{m}'_{-i}(y^*)$ all $m_{-i}, \hat{m}_{-i} \in M^*_{-i}$ by lemma 1 (iv).

But, $1 - \lim_t m'_{-it}(y^*) = \lim_t m'_{it}(y^*)$ and $1 - \lim_n \left[\frac{dm^S_{-it_n}}{dy}(\bar{y})\right] = \lim_n m'_{it_n}(\bar{y}) \Rightarrow \lim_n m''_{it_n}(y) \geq 0$ some $y \in [\bar{y}, y^*]$, a contradiction since $-K_2 \leq m''_{it_n}(y) \leq -K_3 < 0$ all y.

Case II. (i) follows immediately from definition 5(d) and 8(b).

To prove (ii), suppose that $m'_{it_n}(\tilde{y}_{it_n}) \to \tilde{r}_i \neq r_i^* = \lim_t m'_{it}(y_T)$ for some subsequence $\{\tilde{y}_{it_n}\} \subset \{\tilde{y}_{it}\}$, and let $\bar{y} = \lim_n \tilde{y}_{it_n}$. Clearly, $y_T = \lim_t y_t$. Without loss of generality, assume $\bar{y} < y_T$. By definition, $\tilde{r}_i = 1 - \lim_n \left[\frac{dm^S_{-it_n}}{dy}(\tilde{y}_{it_n})\right]$.

Observe that $m'_{-it-1}(y_t) \leq \left[\frac{dm^S_{-it}}{dy}(y_t)\right] < m'_{-it}(y_t)$. $\therefore \lim_t \left[\frac{dm^S_{-it}}{dy}(y_t)\right] = \lim_t m'_{-it}(y_t)$. Hence, $\lim_n \left[\frac{dm^S_{-it_n}}{dy}(\tilde{y}_{it_n}) + m'_{it_n}(\tilde{y}_{it_n})\right] = 1 = \left[\lim_t \frac{dm^S_{-it}}{dy}(y_t) + m'_{it}(y_t)\right]$.

But, this is a contradiction: $\bar{y} < y_T$ and $m''_{it}(y) \leq -K_3 < 0$ all y, t $\Rightarrow \lim_n m'_{it_n}(\bar{y}) > \lim_t m'_{it}(y_T)$; thus, $\lim_n \left[\frac{dm^S_{-it_n}}{dy}(\bar{y})\right] < \lim_t \left[\frac{dm^S_{-it}}{dy}(y_T)\right]$ and it follows that $m^S_{jt} \notin M_{jt}$ for some $j \neq i$ and some t large.

Finally, observe that if $\frac{\partial \eta_i}{\partial m}(M_{-it}, m) \equiv 0$, all t, then $m_{it} = m_{it-1}$ all $t \geq T + 1$.//

FOOTNOTES

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1. For an elegant survey of the mechanisms that have been proposed, see Groves (1979).

2. If consumers' utilities are additive and linear in private goods, then truthtelling is a dominant strategy, and the dynamic question is moot; otherwise, however, truthtelling may not be a best strategy away from equilibrium. See Groves (1979).

3. See Greenberg, Mackay and Tideman (1977), Bamford (1979) and Ledyard (1978).

4. Strategy equilibrium and corresponding resource or real equilibria should be distinguished in these models because there is not, in general, a one-to-one relationship between the two types of equilibria. Agents' response rules are defined on strategy sets distinct from the allocation space, and relate to the allocation space only indirectly through the government administered mechanism. Thus, it is possible to have cyclical behavior in strategies but an unchanging resource allocation.

5. The mechanism less the enforcement rule is examined elsewhere, and one conclusion of these considerations is that the budget cannot be balanced for this mechanism because of an overabundance of functions that satisfy the choice criteria. See Groves (1979).

6. To our knowledge, no mechanism has been suggested that simultaneously elicits "truth" and ensures a balanced budget in equilibrium. The G-L quadratic mechanism does not necessarily elicit truthful information in equilibrium.

7. If one defines $t^i = -z_i$, then t^i is interpreted as the tax paid by consumer i for the public good level y, and the public good balance equation becomes $\sum_{i \leq N} t^i = y$.

8. Because a specific production relation has been assumed, and because there is a single private good, the production relation need not be represented explicitly in a specification of a state of the economy or of the economy itself. It is implicit, however, in W, the set of feasible states.

9. Mechanisms in the literature and the Nash equilibrium concept used for these mechanisms are special cases of definitions 2 and 3. To see this, just define $E(M_o, m) = M_o$, and observe that, for each i, m_i^* is at least as good as any other $m_i \in M_i$ (in terms of the choice rule $S^i(\cdot)$) given the messages m_j^* of agents $j \neq i$.

10. This follows from the fact that $M_{t+1} \subset M_t$ for all $t \geq 1$ implies $\lim\sup_t M_t = \lim\inf_t M_t$ (see Friedman (1970).

11. (1) follows from maximizing a weighted sum of consumers' utilities subject to feasibility and production constraints, a standard method for generating Pareto optimal allocations.

12. This difficulty is also circumvented by the government regardless of the nature of agents' response rules $S(\cdot)$, if it alters the enforcement rule to require that reported marginal valuations for each y_t be non-increasing for all $s > t$. See property (v) of lemma 1. Since the entire problem arises because it is possible to have a real stationary allocation but periodicity in messages, other stopping rules that focus on the real allocation can be used to prevent infinite cycling. However, efficiency cannot then be guaranteed.

13. See Greenberg, Mackay and Tideman (1977).

14. There are specifications of the tax rules in which the balance term for any i is a function only of the previous messages of other agents $j \neq i$ that yield also an asymptotically balanced budget and no consumer bankruptcy when used in conjunction with the enforcement rule given here. Such rules are easily constructed.

15. Consider the following example: Let $X = C(I,I)$ where $I = [0,1]$, and let $X_f = \{ x \in I: f(x) = \max_{y \in I} f(y) \}$, a non-empty, closed set for each $f \in X$.

Define $x_f = \min X_f$ for each $f \in X$, and $f_\varepsilon(x) = 1/2 + \varepsilon x$, $\varepsilon < 1/2$. Then, $f_\varepsilon \to f_o \equiv 1/2$ as $\varepsilon \to 0$ whereas $x_{f_\varepsilon} = 1$ does not converge to $x_{f_o} = 0$.

16. See Groves (1979) for a discussion of a statement of the incentive compatibility properties of static formulations of demand revelation mechanisms similar to that developed dynamically in this paper.

17. See Robinson-Suchanek (1979) for a generalized static statement and development of this relationship.

18. Introduction of the correspondence $\eta_i(\cdot)$ is necessary to deal with response rules that are dependent only on other consumers' sets of permissable messages, thus being independent of their previous communication except as reflected by the enforcement rule in the restrictions on future communications.

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